Decomposition of Viscosity Tensors for Gyrotropic Media

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For a medium whose anisotropy is caused by a physical quantity expressed by an axial vector, e. g. an ambient magnetic field, the viscosity tensor has 8 independent components and is consequently decomposed in 8 terms. One term, multiplied with the bulk viscosity, connects the traces of the viscous pressure tensor and the velocity gradient, while traces and tracefree parts of these two tensors are coupled by two other terms, whose coefficients are required to be equal by Onsager's symmetry relations. The remaining 5 terms are represented by generalized Hess-Waldmann projectors. The corresponding 5 viscosity coefficients are related to those introduced by Braginskii, de Groot-Mazur, and Hess-Waldmann.

1. General Media

A fourth rank viscosity tensor $\overleftrightarrow{\boldsymbol{\eta}}$, connecting the symmetric parts of the viscous pressure tensor $\mathbf{p}$ and of the velocity gradient $\nabla \boldsymbol{v}$ as

$$\widehat{\mathbf{p}} = -2\,\overleftrightarrow{\boldsymbol{\eta}} : \widehat{\nabla \boldsymbol{v}}\,, \tag{1}$$

has 36 independent components. It must be symmetric with respect to the first pair of indices as well as with respect to the second pair. If the two (symmetric) second order tensors $\widehat{\mathbf{p}}$ and $\widehat{\nabla \boldsymbol{v}}$ are decomposed as

$$\widehat{\mathbf{p}} = \overline{\mathbf{p}} + \text{trace}\ \mathbf{p}\,\mathbf{I}/3$$
$$\widehat{\nabla \boldsymbol{v}} = \overline{\nabla \boldsymbol{v}} + \text{div}\ \boldsymbol{v}\,\mathbf{I}/3 \tag{2}$$

into tracefree (irreducible) parts $\overline{\mathbf{p}}, \overline{\nabla \boldsymbol{v}}$ (indicating derivations from isotropy) and isotropic parts, the fourth order viscosity tensor is correspondingly decomposed with the reduction operator

$$\overleftrightarrow{\mathbf{R}} := \overleftrightarrow{\mathbf{I}} - \tfrac{1}{3}\mathbf{I}\,\mathbf{I} \tag{3}$$

into four parts:

$$\overleftrightarrow{\boldsymbol{\eta}} = \overleftrightarrow{\mathbf{R}} : \overleftrightarrow{\boldsymbol{\eta}} : \overleftrightarrow{\mathbf{R}} + \overleftrightarrow{\mathbf{R}} : \overleftrightarrow{\boldsymbol{\eta}} : \frac{\mathbf{I}}{3}\mathbf{I}$$
$$+ \mathbf{I}\frac{\mathbf{I}}{3} : \overleftrightarrow{\boldsymbol{\eta}} : \overleftrightarrow{\mathbf{R}} - \mathbf{I}\frac{\mathbf{I}}{3} : \overleftrightarrow{\boldsymbol{\eta}} : \frac{\mathbf{I}}{3}\mathbf{I}\,. \tag{4}$$

With $\overleftrightarrow{\mathbf{I}}$ as four dimensional unit tensor the reduction operator $\overleftrightarrow{\mathbf{R}}$ (3) projects any second rank tensor into the subspace of tracefree tensors. The scalar

$$\eta_{\mathrm{v}} := \mathbf{I}/3 : (-2\,\overleftrightarrow{\boldsymbol{\eta}}) : \mathbf{I}/3 \tag{5}$$

in the last term of (4) is the bulk viscosity, connecting trace $\mathbf{p}/3$ with div $\boldsymbol{v}$ (since $\mathbf{I} : \mathbf{I} =$ trace $\mathbf{I} = 3$). The other 35 independent components of $\overleftrightarrow{\boldsymbol{\eta}}$ must be contained in the first three terms of (4). The second and third term couple tracefree parts with isotropic parts.

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2. Gyrotropic Media

In the following it is assumed that the medium is gyrotropic, i. e. its anisotropy is caused by a physical quantity to be represented by an axial vector, e. g. an ambient magnetic field $\boldsymbol{B}$. Then the viscosity tensor $\overleftrightarrow{\boldsymbol{\eta}}$ must be rotationally invariant around the direction $\widehat{\boldsymbol{B}}$. This reduces the number of independent components to eight [1], seven of them must be contained in the first three terms of (4).

The second order tensors

$$\overleftrightarrow{\mathbf{R}} : \overleftrightarrow{\boldsymbol{\eta}} : \mathbf{I}/3 \quad \text{and} \quad \mathbf{I}/3 : \overleftrightarrow{\boldsymbol{\eta}} : \overleftrightarrow{\mathbf{R}}$$

[in the second and third term of (4)] must be tracefree because of the operator $\overleftrightarrow{\mathbf{R}}$ (3). They must be composed of the vector $\widehat{\boldsymbol{B}}$ (and of the unit tensor $\mathbf{I}$), since no other direction is distinguished. Hence they are both proportional to $\widehat{\boldsymbol{B}}\,\widehat{\boldsymbol{B}} - \mathbf{I}/3$ and we write

$$\overleftrightarrow{\mathbf{R}} : \overleftrightarrow{\boldsymbol{\eta}} : \mathbf{I}/3 = -\frac{\xi}{2}\,(\widehat{\boldsymbol{B}}\,\widehat{\boldsymbol{B}} - \mathbf{I}/3)$$
$$\mathbf{I}/3 : \overleftrightarrow{\boldsymbol{\eta}} : \overleftrightarrow{\mathbf{R}} = -\frac{\zeta}{2}\,(\widehat{\boldsymbol{B}}\,\widehat{\boldsymbol{B}} - \mathbf{I}/3)\,. \tag{6}$$

Thus the first term $\overleftrightarrow{\mathbf{R}} : \overleftrightarrow{\boldsymbol{\eta}} : \overleftrightarrow{\mathbf{R}}$ in (4) must contain only 5 independent components.

Since the reduction operator $\overleftrightarrow{\mathbf{R}}$ (3) is a projector, it must be possible to decompose $\overleftrightarrow{\mathbf{R}} : \overleftrightarrow{\boldsymbol{\eta}} : \overleftrightarrow{\mathbf{R}}$ as

$$\overleftrightarrow{\mathbf{R}} : \overleftrightarrow{\boldsymbol{\eta}} : \overleftrightarrow{\mathbf{R}} = \sum_{k=-2}^{+2} \eta_k\,\overleftrightarrow{\mathbf{B}}_k \tag{7}$$

with 5 orthogonal projectors $\overleftrightarrow{\mathbf{B}}_k$ ($k=0, \pm 1, \pm 2$), the sum of which is

$$\sum_{k=-2}^{+2} \overleftrightarrow{\mathbf{B}}_k = \tfrac{1}{2}(\overleftrightarrow{\mathbf{I}} + \tilde{\overleftrightarrow{\mathbf{I}}}) - \tfrac{1}{3}\mathbf{I}\,\mathbf{I}\,, \tag{8}$$

i.e. the projector into the subspace of symmetric traceless second rank tensors. The tilde means the interchange of the last two indices.

The set of 5 orthogonal fourth rank projectors $\overleftrightarrow{\mathbf{B}}_k$ is represented as follows:

With the set of 3 orthogonal second rank projectors

$$\mathbf{P}_0 := \hat{\boldsymbol{B}}\,\hat{\boldsymbol{B}}\,, \quad \mathbf{P}_{\pm 1} := \tfrac{1}{2}[\mathbf{I} - \hat{\boldsymbol{B}}\,\hat{\boldsymbol{B}} \pm \mathrm{i}\,\hat{\boldsymbol{B}} \times \mathbf{I}] = \mathbf{P}^*_{\mp 1} \tag{9}$$

the set of 5 fourth rank tensors ($k_{1,2} = 0, \pm 1, \pm 2$)

$$\overleftrightarrow{\mathbf{P}}_k := \sum_{k_1}\sum_{k_2} \delta_{k,k_1+k_2} \widetilde{\{\mathbf{P}_{k_1}\mathbf{P}_{k_2}\}} = \overleftrightarrow{\mathbf{P}}^*_{-k}\,, \tag{10}$$

with

$$\overleftrightarrow{\mathbf{P}}_k : \mathbf{I} = \mathbf{I} : \overleftrightarrow{\mathbf{P}}_k = \delta_{0k}\,\mathbf{I}\,, \tag{11}$$

has been shown by Hess and Waldmann[2] to be a set of orthogonal projectors with

$$\sum_{k=-2}^{+2} \overleftrightarrow{\mathbf{P}}_k = \overleftrightarrow{\mathbf{I}}\,. \tag{12}$$

The braces in (10) mean the interchange of the inner two indices. With (11) and the orthogonality of the $\overleftrightarrow{\mathbf{P}}_k$ it can be shown[3] that

$$\overleftrightarrow{\mathbf{B}}_k := \tfrac{1}{2}(\overleftrightarrow{\mathbf{P}}_k + \tilde{\overleftrightarrow{\mathbf{P}}}_k) - \frac{\delta_{0k}}{3}\mathbf{I}\,\mathbf{I} = \overleftrightarrow{\mathbf{B}}^*_{-k} \tag{13}$$

is a set of orthogonal projectors with the property (8).

Combining (4), (5), (6), (7) the viscosity relation for a gyrotropic medium can be written as

$$\hat{\mathbf{p}} = -2\,\overleftrightarrow{\boldsymbol{\eta}} : \widehat{\boldsymbol{\nabla} \boldsymbol{v}}$$

with

$$\overleftrightarrow{\boldsymbol{\eta}} = \sum_{k=-2}^{+2} \eta_k \overleftrightarrow{\mathbf{B}}_k + \frac{1}{2}\eta_{\mathrm{v}}\,\mathbf{I}\,\mathbf{I} - \frac{1}{2}\xi\left(\hat{\boldsymbol{B}}\,\hat{\boldsymbol{B}} - \frac{\mathbf{I}}{3}\right)\mathbf{I} - \frac{1}{2}\zeta\,\mathbf{I}\left(\hat{\boldsymbol{B}}\,\hat{\boldsymbol{B}} - \frac{\mathbf{I}}{3}\right). \tag{14}$$

3. Onsager Relations

Inspection of the definition (10) for the Hess-Waldmann projectors $\overleftrightarrow{\mathbf{P}}_k$ shows that they satisfy the Onsager symmetry relations. Hence the projectors $\overleftrightarrow{\mathbf{B}}_k$ (13) do so, too, as well as the coefficient $\mathbf{I}\,\mathbf{I}$ of the bulk viscosity in (14). Only the last two terms in (14) are affected by the requirements of the Onsager relations, which lead to

$$\xi = \zeta\,. \tag{15}$$

4. Representations of Braginskii, de Groot-Mazur and Hess-Waldmann

Since the projectors $\mathbf{P}_k = \mathbf{P}^*_{-k}$ (9), $\overleftrightarrow{\mathbf{P}}_k = \overleftrightarrow{\mathbf{P}}^*_{-k}$ (10) and $\overleftrightarrow{\mathbf{B}}_k = \overleftrightarrow{\mathbf{B}}^*_{-k}$ (13) are complex quantities, it is useful for the representation of measurements to rearrange the doubly anisotropic part $\Sigma\,\eta_k \overleftrightarrow{\mathbf{B}}_k$ of $\overleftrightarrow{\boldsymbol{\eta}}$ (14) as

$$\begin{aligned}\sum_{k=-2}^{+2} \eta_k \overleftrightarrow{\mathbf{B}}_k &= \eta_0 \overleftrightarrow{\mathbf{B}}_0 + \sum_{k=1}^{2} \frac{\eta_k + \eta_{-k}}{2}(\overleftrightarrow{\mathbf{B}}_k + \overleftrightarrow{\mathbf{B}}_{-k}) + \sum_{k=1}^{2} i\,\frac{\eta_k - \eta_{-k}}{2}\,\frac{\overleftrightarrow{\mathbf{B}}_k - \overleftrightarrow{\mathbf{B}}_{-k}}{\mathrm{i}} \\ &= \eta_0 \overleftrightarrow{\mathbf{B}}_0 + \sum_{k=1}^{2} \eta_{k+}\, 2\,\mathrm{Re}\,\overleftrightarrow{\mathbf{B}}_k + \sum_{k=1}^{2} i\,\eta_{k-}\, 2\,\mathrm{Im}\,\overleftrightarrow{\mathbf{B}}_k\end{aligned} \tag{16}$$

with

$$\eta_{k\pm} := \tfrac{1}{2}(\eta_k \pm \eta_{-k})\,. \tag{17}$$

This representation has been used by Braginskii[4] with the following notation:

$$\eta_0^{\mathrm{Bra}} = \eta_0\,, \quad \eta_1^{\mathrm{Bra}} = \eta_{2+}\,, \quad \eta_2^{\mathrm{Bra}} = \eta_{1+}\,, \quad \eta_3^{\mathrm{Bra}} = -i\,\eta_{2-}\,, \quad \eta_4^{\mathrm{Bra}} = -i\,\eta_{1-}\,. \tag{18}$$

The first and third term at the right-hand side of (16) have been combined by de Groot-Mazur[1] as follows:

$$\eta_0 \overleftrightarrow{\mathbf{B}}_0 + \eta_{2+}\, 2\,\mathrm{Re}\,\overleftrightarrow{\mathbf{B}}_2 = \eta_0(\overleftrightarrow{\mathbf{B}}_0 - 2\,\mathrm{Re}\,\overleftrightarrow{\mathbf{B}}_2) + \tfrac{1}{2}(\eta_0 + \eta_{2+})\, 4\,\mathrm{Re}\,\overleftrightarrow{\mathbf{B}}_2\,. \tag{19}$$

This has led to the following notation:

$$\eta_1^{\mathrm{Gr\,Ma}} = \eta_0\,, \quad \eta_2^{\mathrm{Gr\,Ma}} = \tfrac{1}{2}(\eta_0 + \eta_{2+})\,, \quad \eta_3^{\mathrm{Gr\,Ma}} = \eta_{1+}\,, \quad \eta_4^{\mathrm{Gr\,Ma}} = i\,\eta_{2-}\,, \quad \eta_5^{\mathrm{Gr\,Ma}} = i\,\eta_{1-}\,. \tag{20}$$

Since Hess and Waldmann[2] introduced the second rank projectors $\mathbf{P}_k$ (9) as $\mathbf{P}^{(-k)}$, their viscosity coefficients $\eta_k^{\mathrm{He\,Wa}}$ are related to the coefficients η_k (14) of the present paper by

$$\eta_k^{\mathrm{He\,Wa}} = \eta_{-k}\,. \tag{21}$$

5. Viscosity Tensor Connecting Tracefree Tensors

If the viscosity tensor $\overleftrightarrow{\boldsymbol{\eta}}$ is required to connect only the tracefree parts $\overline{\mathbf{p}}$ and $\overline{\boldsymbol{\nabla} \boldsymbol{v}}$ as

$$\overline{\mathbf{p}} = -2\,\overleftrightarrow{\boldsymbol{\eta}} : \overline{\boldsymbol{\nabla} \boldsymbol{v}} \tag{22}$$

then it has in general only 25 independent components. They must be all contained in the first term $\overleftrightarrow{\mathbf{R}} : \overleftrightarrow{\eta} : \overleftrightarrow{\mathbf{R}}$ of (4), since the reduction operator $\overleftrightarrow{\mathbf{R}}$ (3) must not have any influence on this $\overleftrightarrow{\eta}$. Thus η_v (5), ξ and ζ (6) are zero in this case. With the additional requirement to be rotationally invariant around the direction $\hat{\boldsymbol{B}}$ there remain only the 5 independent components η_k (7). The projectors $\overleftrightarrow{\mathbf{B}}_k$ (13), if acting on symmetric traceless tensors $\overline{\nabla \boldsymbol{v}}$, can be replaced by the projectors $\overleftrightarrow{\mathbf{P}}_k$ (10) in this case. Since their sum $\Sigma \overleftrightarrow{\mathbf{P}}_k$ yields the unit tensor $\mathbf{I}$ (12), the 5 components η_k are the eigenvalues of $\overleftrightarrow{\eta}$ in this case [2].

[1] S. R. de Groot and P. Mazur, Non-Equilibrium Thermodynamics; North-Holland, Amsterdam 1962, Chapter XII, § 2.

[2] S. Hess and L. Waldmann, Kinetic Theory for a Dilute Gas of Particles with Spin; Z. Naturforsch. **26 a**, 1057 [1971], Appendix A2.

[3] U. Sturhann, Darstellungen des Viskositätstensors in magnetisierbaren Medien; Diplom-Thesis Düsseldorf 1976, Eqs. (2.27) and (2.42).

[4] S. I. Braginskii, Transport Processes in a Plasma; Review of Plasma Physics **1**, 205 [1965], Sect. 4.